The spectrum of a quantum Lifshitz black hole in two dimensions

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Four-dimensional Lifshitz black hole

We consider the following four-dimensional action obtained from a consistent truncation of heterotic string theory [1, 2, 3]:

$$I=\int d^4x\sqrt{-g}\,e^{-2\phi}\left[R+4(\nabla\phi)^2-\tfrac{2}{3}(\nabla\psi)^2-F^2-e^{2(\phi-\frac{q}{3}\psi)}F^2\right], \tag{1}$$

where ϕ is the dilaton, $\psi=\frac{3}{q}\phi+const$ is a compact scalar, and $F_{\mu\nu}$ is the Maxwell field strength. The structure of this theory is similar in spirit to the Einstein–Maxwell–Dilaton actions studied in the context of Lifshitz holography by Marika Taylor [4, 5], and can be mapped to that form via a Weyl rescaling to the Einstein frame. A magnetically charged black hole solution is given by:

$$\begin{aligned}ds^2&=-\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{-1+\alpha}{1+\alpha}}dt^2+\frac{dr^2}{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)}+r^2d\Omega_2^2,\\ e^{-2\phi}&=\left(1-\frac{r_-}{r}\right)^{\frac{1}{1+\alpha}},\quad F_{\mu\nu}=\frac{Q_m}{r^2}\epsilon_{\mu\nu},\quad \alpha=\frac{3}{2q^2}.\end{aligned} \tag{2}$$

The black hole mass and magnetic charge are related to $r_\pm$ by

$$M=\frac{1}{2}r_++\frac{1}{2}\frac{\alpha}{1+\alpha}r_-,\qquad Q_m^2=\frac{1}{2}\frac{r_+r_-}{1+\alpha}. \tag{3}$$

The Hawking temperature is given by

$$T=\frac{1}{4\pi r_+}\left(1-\frac{r_-}{r_+}\right)^{\frac{\alpha}{1+\alpha}}, \tag{4}$$

so extremality occurs when $r_+=r_-=r_0$.

Near-extremal decoupling limit

We work in the canonical ensemble by fixing the magnetic charge to its extremal value, $Q_0=\frac{r_0}{\sqrt{2(1+\alpha)}}$. The horizons then seperate at lowest order in temperature as

$$r_\pm(T)=r_0\pm\tfrac{r_0}{2}(4\pi r_0T)^{1+\frac{1}{\alpha}}+\dots, \tag{5}$$

and the mass near extremality becomes

$$M(T)=M_0+\frac{T^{1+\frac{1}{\alpha}}}{M_{\text{gap}}}, \tag{6}$$

where the extremal mass and the mass gap are respectively

$$M_0=\frac{1+2\alpha}{1+\alpha}\frac{r_0}{2},\qquad \frac{1}{M_{\text{gap}}}=\frac{1}{1+\alpha}\frac{r_0}{4}(4\pi r_0)^{1+\frac{1}{\alpha}}. \tag{7}$$

To isolate the infrared geometry, we take a near-extremal decoupling limit using the rescaled coordinates

$$t=\frac{\tilde t}{2\pi T},\qquad r=r_++r_0\left[\left(\frac{\tilde r}{\tilde r_h}\right)^{1+\frac{1}{\alpha}}-1\right](4\pi r_0T)^{1+\frac{1}{\alpha}}, \tag{8}$$

where the horizon location in the new radial coordinate is given by

$$\tilde r_h=2r_0^2\left(1+\tfrac{1}{\alpha}\right). \tag{9}$$

The metric then takes the form

$$ds^2=-F(\tilde r)\,d\tilde t^2+\frac{d\tilde r^2}{F(\tilde r)}+r_0^2d\Omega_2^2+\mathcal{O}(T^{1+\frac{1}{\alpha}}), \tag{10}$$

with

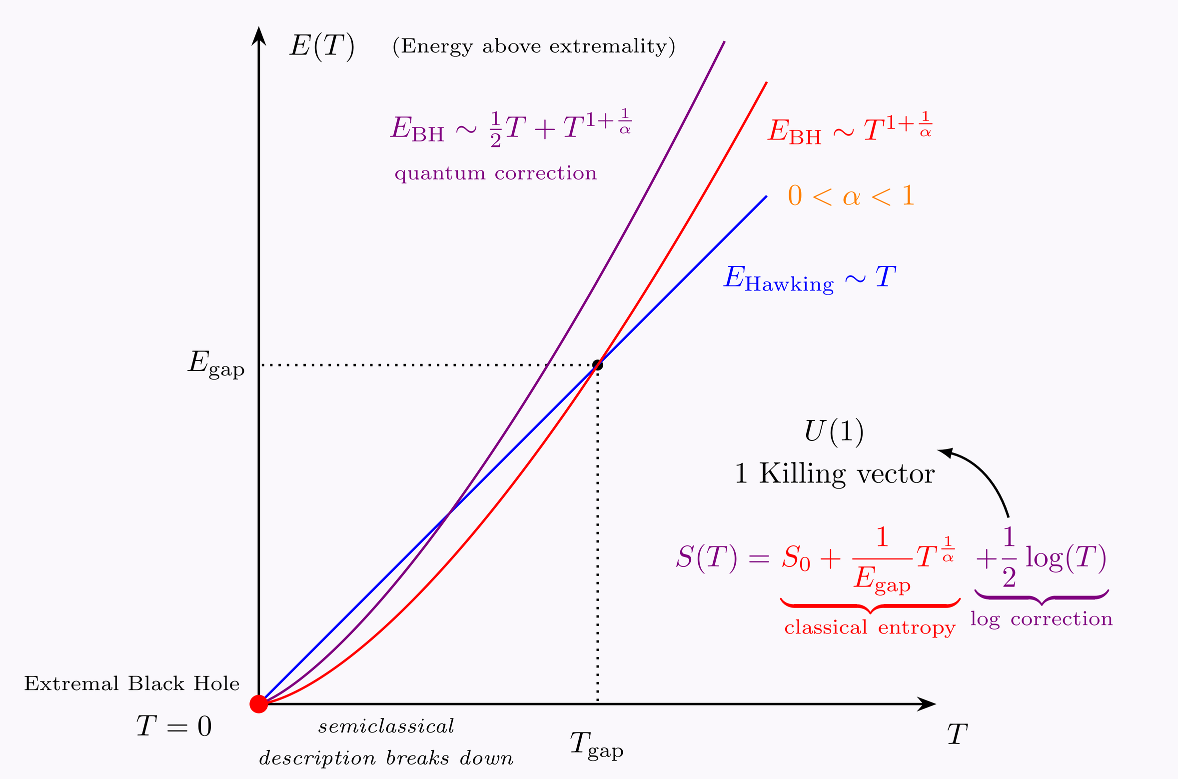
$$F(\tilde r)=\frac{2\alpha\tilde r_h}{1+\alpha}\left[\left(\frac{\tilde r}{\tilde r_h}\right)^2-\left(\frac{\tilde r}{\tilde r_h}\right)^{1-\frac{1}{\alpha}}\right]. \tag{11}$$

Mass gap and quantum corrections

At low temperatures, the energy above extremality scales as

$$E_{\text{BH}}\sim T^{1+\frac{1}{\alpha}},\qquad E_{\text{Hawking}}\sim T. \tag{12}$$

This leads to a **mass gap puzzle** [6]. At low temperatures, the black hole lacks the energy to radiate even a single Hawking quantum. This signals a breakdown of the semiclassical picture. Quantum corrections can resolve the discrepancy. A $\sqrt{T}$ prefactor in the one loop partition function adds a linear term to the energy and lowers the gap. This corresponds to a $\frac{1}{2}\log T$ entropy correction rather than the $\frac{3}{2}\log T$ seen in JT gravity, reflecting the reduced symmetry: Only one global Killing vector survives, in contrast to the three of the $SL(2,\mathbb{R})$ invariant JT model.



Two dimensional Lifshitz dilaton model

Starting from the four-dimensional action (1), a near-horizon dimensional reduction leads to an effective two dimensional dilaton gravity model [1, 7]

$$I_{\text{Lif}}(\alpha)=-\tfrac{1}{2}\int_{\mathcal{M}}d^2x\sqrt{g}\,e^{-2\phi}\Big(R+4(1-\alpha)(\nabla\phi)^2+4\lambda^2\Big). \tag{13}$$

This model interpolates between well-known cases:

- $\alpha=1$: Jackiw-Teitelboim (JT) [8, 9].
- $\alpha=0$: Callan-Giddings-Harvey-Strominger (CGHS) [10].

The equations of motion for this dilaton theory are

$$\nabla_\mu\nabla_\nu\phi=2\alpha\,\nabla_\mu\phi\,\nabla_\nu\phi+\left(\nabla^2\phi-(1+\alpha)(\nabla\phi)^2+\lambda^2\right)g_{\mu\nu}, \tag{14a}$$

$$R+4(1-\alpha)\nabla^2\phi-4(1-\alpha)(\nabla\phi)^2+4\lambda^2=0. \tag{14b}$$

The saddle point solution for the metric and the dilaton in Schwarzschild gauge, $ds^2=f(r)dt^2+\frac{dr^2}{f(r)}$, is given by

$$f(r)=(ar)^2-b(ar)^{1-\frac{1}{\alpha}},\qquad e^{-2\phi}=(ar)^{\frac{1}{\alpha}}, \tag{15}$$

where b is a free parameter and $a=\frac{2\alpha\lambda}{\sqrt{1+\alpha}}$ is a convenient redefinition of λ. This geometry is asymptotically AdS_2. The model admits only a single global Killing vector, ∂_t.

Thermodynamics

The thermodynamical properties of the saddle point solution have previously been analysed in [2, 11, 12]. Notably the Hawking temperature is given by

$$T=\frac{f'(r_h)}{4\pi}=\frac{1}{4\pi}\left(1+\tfrac{1}{\alpha}\right)ab^{\frac{\alpha}{1+\alpha}}, \tag{16}$$

where the horizon radius is $ar_h=b^{\frac{\alpha}{1+\alpha}}$. The Bekenstein-Hawking entropy is given by

$$S_{\text{BH}}(T)=(1+\alpha)C\,T^{\frac{1}{\alpha}},\qquad C=\frac{a}{2\alpha}\left(\frac{4\pi}{a(1+\frac{1}{\alpha})}\right)^{1+\frac{1}{\alpha}}. \tag{17}$$

This scaling relation between temperature and entropy is a hallmark feature of models with Lifshitz-scaling. Additionally there is a nice two dimensional incarnation of the Smarr formula

$$T\,S=(1+\alpha)E,\qquad E=\frac{ab}{2\alpha}. \tag{18}$$

where E denotes the energy or ADM mass.

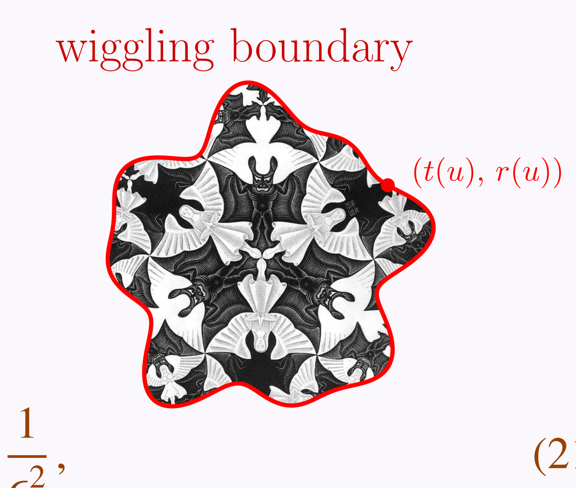
Boundary terms and wiggling boundary action

We add the Gibbons-Hawking-York (I_{GHY}) term to make the variational problem well defined and a counter term (I_{counter}) which renders the Euclidean on-shell action finite

$$I_{\text{GHY}}+I_{\text{counter}}=-\int_{\partial\mathcal{M}}dt\,\sqrt{h}\,e^{-2\phi}\left(K-\frac{a}{\alpha}\right). \tag{19}$$

Here K is the extrinsic scalar curvature, h is the determinant of the induced metric on the boundary and n^μ is the unit normal vector on the boundary. By making use of the equations of motion the bulk piece can be rewritten as a boundary term and hence the entire action can be written in the following form

$$I_{\text{Lif}}=-\int_{\partial\mathcal{M}}dt\,\sqrt{h}\,e^{-2\phi}\left(-2(1-\alpha)n^\mu\nabla_\mu\phi+K-\frac{a}{\alpha}\right). \tag{20}$$



Wiggling boundary ansatz:
We parametrize the boundary curve by $(t(u),r(u))$ and impose a fixed length of the boundary with the wiggling boundary constraint

$$h_{uu}=\frac{1}{\epsilon^2}, \tag{21}$$

this is solved by

$$r(u)=\frac{1}{a\,\epsilon\,t'(u)}+\mathcal{O}(\epsilon^3). \tag{22}$$

To obtain a finite boundary action, we rescale

$$E\to\epsilon^{1-\alpha}E,\qquad t(u)\to\epsilon^{\alpha-1}t(u). \tag{23}$$

We now compute the geometric quantities:

$$\begin{aligned}K&=a+\tfrac{1}{a}\left(\text{Sch}(t,u)+aE\,t'(u)^{1+\frac{1}{\alpha}}\right)\epsilon^2+\mathcal{O}(\epsilon^4),\\ \sqrt{h}&=\tfrac{1}{\epsilon}+\mathcal{O}(\epsilon),\\ e^{-2\phi}&=\tfrac{1}{t'(u)^{1/\alpha}}\tfrac{1}{\epsilon}+\mathcal{O}(\epsilon^3),\\ n^\mu\nabla_\mu\phi&=-\tfrac{a}{2\alpha}+\tfrac{1}{2}\left(Et'(u)^{1+\frac{1}{\alpha}}+\tfrac{1}{2a\alpha}\left(\tfrac{t''(u)}{t'(u)}\right)^2\right)\epsilon^2+\mathcal{O}(\epsilon^4).\end{aligned} \tag{24}$$

Inserting these into the action and taking the limit $\epsilon\to0$ yields the **deformed Schwarzian** action:

$$I_{\text{dSch}}=-\frac{1}{a}\int\frac{du}{t'(u)^{\frac{1}{\alpha}}}\left(\text{Sch}(t,u)-\tfrac{1-\alpha}{2\alpha}\left(\frac{t''(u)}{t'(u)}\right)^2+\alpha aEt'(u)^{1+\frac{1}{\alpha}}\right). \tag{25}$$

Partition functions and logarithmic correction

We approximate the Lifshitz partition function around the deformed Schwarzian saddle as follows

$$Z_{\text{Lif}}=\int[\mathcal{D}g_{ab}][\mathcal{D}\phi]\,e^{-I_{\text{Lif}}[g_{ab},\phi]}\approx\int_{\text{Diff}(S^1)/U(1)}[\mathcal{D}t]_{\text{Lif}}\,e^{-I_{\text{dSch}}[t]}. \tag{26}$$

Expanding around the saddle $t(u)=u+\epsilon(u)$, and decomposing in a Fourier series $\epsilon(u)=\sum_{n\neq0}\left(\epsilon_n^{(R)}+i\epsilon_n^{(I)}\right)e^{-\frac{2\pi}{\beta}inu}$ the quadratic action is given to second order in fluctuations as

$$I_{\text{dSch}}=-\alpha C\beta^{-1/\alpha}-\frac{2\pi^2}{\beta\alpha a}\sum_{n\neq0}n^4\left(\epsilon_n^{(R)}\epsilon_n^{(R)}+\epsilon_n^{(I)}\epsilon_n^{(I)}\right), \tag{27}$$

along with the measure

$$[\mathcal{D}t]_{\text{Lif}}=\prod_{n\geq1}4\pi n\,d\epsilon_n^{(R)}d\epsilon_n^{(I)}. \tag{28}$$

Note for comparison the measure for JT is [13, 14, 15]

$$[\mathcal{D}t]_{\text{JT}}=\prod_{n\geq2}4\pi n(n^2-1)\,d\epsilon_n^{(R)}d\epsilon_n^{(I)}. \tag{29}$$

where $n=\pm1$ correspond to symmetries generated by the two additional Killing vectors of AdS_2. The one-loop integral is then

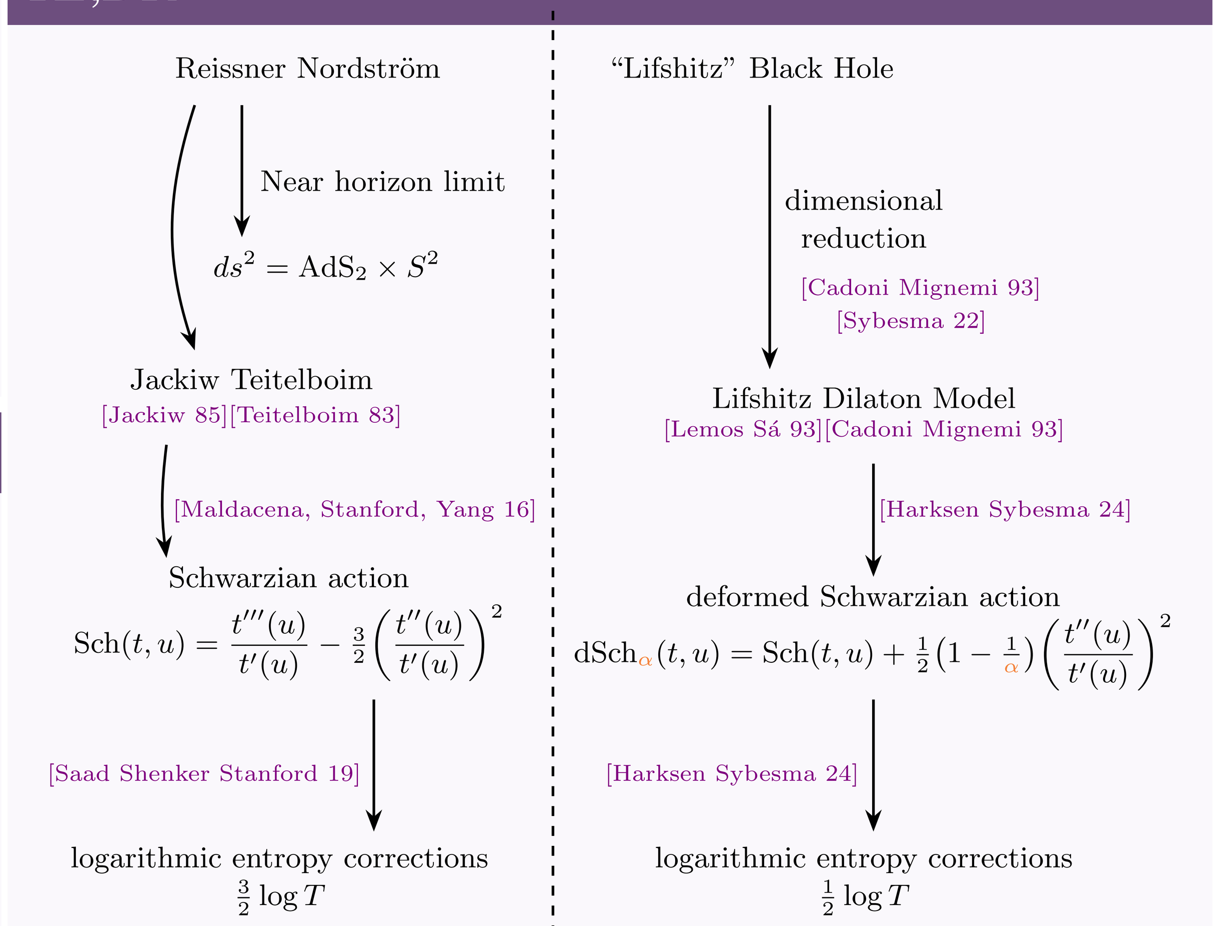
$$\begin{aligned}Z_Q(\beta)&=e^{\alpha C\beta^{-1/\alpha}}\prod_{n\geq1}4\pi n\int_0^\beta d\epsilon_n^{(R)}d\epsilon_n^{(I)}e^{-\frac{2\pi^2n^4}{\beta a\alpha}\left(\left(\epsilon_n^{(R)}\right)^2+\left(\epsilon_n^{(I)}\right)^2\right)}\\ &=e^{\alpha C\beta^{-1/\alpha}}\prod_{n\geq1}\frac{2\alpha\beta a}{n^3}=\frac{1}{4\pi^{3/2}\sqrt{\alpha a\beta}}e^{\alpha C\beta^{-1/\alpha}}.\end{aligned} \tag{30}$$

where in the last step we used **zeta function regularization**. Thus the logarithmic correction to the entropy is given by

$$S(T)=S_0+(1+\alpha)C\,T^{1/\alpha}+\tfrac{1}{2}\log T, \tag{31}$$

where the $\frac{1}{2}\log T$ is the one-loop quantum correction.

TL;DR



We study a one parameter family of two dimensional dilaton models whose entropy scales as $S\propto T^{1/\alpha}$. The boundary dynamics are governed by a deformed Schwarzian action with reduced $U(1)$ symmetry, leading to a $\frac{1}{2}\log T$ correction to the entropy in contrast to the $\frac{3}{2}\log T$ correction of the $SL(2,\mathbb{R})$ invariant JT model.

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